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Introduction to Non Life Mathematics with Reference to Pricing: Concepts, Origin of the Models and a Proposal for an Ongoing Monitoring of the Personalized Premium

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Abstract: *Pricing is a fundamental topic for a non-life insurance company. It includes several underwriting activities to set actuarial models. In the past, it was a fundamental part of risk theory and non-life insurance mathematic and nowadays results in a vast actuarial research field. Since the second half of twentieth century, many papers have been published in actuarial journals. This work originates from the experience of introductory university lectures of the author. After an historical recognition, a proposal is made of a simplified approach to monitor consistency and quality of the personalized premiums over time.*

Keywords: *Pricing; Risks classification; Tariff; Stochastic models; Actuaries.*

JEL Classification Numbers: *C13, G22, M40.*

1. INTRODUCTION

Insurance companies are characterized by the inversion of the production cycle, the length of the settlement processes and the investment of funds. This is due to the activity typically carried out: insuring against a certain event means protecting or defending assets from the negative effects caused by the event. The function and purpose of the insurer can be traced precisely in the offer of capital guarantees for policyholders, through future and eventual services or reimbursement.

Here we offer an overview of the methods of calculating the average or customized premiums that the contracting parties must pay to the

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* *Remark that this article reflects the personal view of the author and not necessarily that of his Institution.*

companies, - prior to the occurrence of the claim event causing the damage - or on the process defined as insurance pricing which for decades has constituted the core of the non-life insurance mathematics. The temporal interconnection is well explained by the following synoptic scheme - selected from the lessons of the Norwegian actuary Haavardsson - in which the period of validity of the contract implies the coverage of claims occurring in the same period.

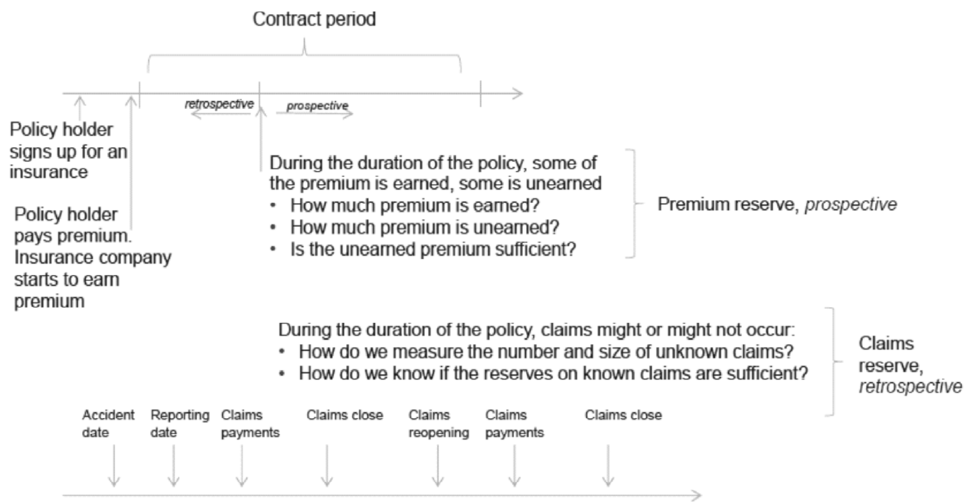

Figure 1: Claims Coverage

2. INSURANCE *PRICING* HISTORY

2.1. Main Items

Numerous studies, particularly on the precautionary principle, show that individuals want to live in a safe society; this feeling towards uncertainty and fear leads people to show special attention to the advantages of safety.

In a civilization more aware of risk, the demand for insurance grows, receiving in response the guarantee of financial security against possible losses. The development of insurance is linked to the persistent need to protect people and their assets against the risks they face. The main role of insurance is to provide means for the transfer (total or partial) of the economic impact deriving from these uncertain events, against the payment of a certain premium.

The probabilistic nature of the risks and their quantification have led to the construction of actuarial science, which is based on probability theory and statistics. Therefore, the task of risk assessment is mainly entrusted to the actuaries who have developed various models over time through which they try to establish a link between the occurrence of risks and the need to know how they occur (frequency and cost of claims). Econometric modelling is used to describe this connection, to determine the probability of events, to assess their economic impact on the insurance company, and to determine insurance premiums that reflect the severity of the risks.

The need to apply different premiums or tariffs according to the degree of risk originates not only from objectively detectable characteristics, but also from the presence of heterogeneity within the insurance portfolio, which also implies the asymmetrical information.

This means that the effect of applying the same premium to the entire portfolio involves the acquisition of adverse risks (at a lower price than their real price) and, on the contrary, discourages the insurance of average o low risks. This scenario can have a spiral effect, which means that the insurer can keep a disproportionate number of "bad" risks in the portfolio and, as a result, must continually increase the insurance premium. This result of the non-existence of a single equilibrium prize under competitive conditions is due to Rothschild and Stiglitz [15] (who received the Nobel Prize for Economics in 2001, especially for 1976 work).

Information problems between the insurance company and the policyholders arise when the insurer has difficulty in understanding the level of risk of the policyholder. The economic literature presents two aspects of information asymmetry, namely adverse selection and moral hazard. Denuit, Maréchal, Pitrebois and Walhin in a 2007 essay [6] believe that adverse selection occurs when policyholders have a better understanding of their claims behaviour than the insurer and take advantage of unknown information. Chiappori, Jullien, Salanié and Salanié (2006) [5] stressed the fact that moral hazard arises when the individual claim frequency of one risk depends on the behaviour of the insured and his decisions in driving. The difference between the two observed phenomena has also been dealt with by Dionne, Michaud and Pinquet (2013) [8], who argue that adverse selection is the effect of unnoticed differences between individuals that influence the optimality of insurance contracting while the moral hazard is the effect after the conclusion of contracts on the unnoticed behaviour of individuals.

The procedure underlying the calculation of a diversified premium is represented by a pricing process that involves several steps. Hence, the acceptance of risk by the insurance company is preceded by a priori analysis that indicates the segmentation or classification of all risks in terms of influence factors, so that each group with identical risks will have the same premium. In this phase of the analysis, the actuary determines the impact of the observable factors on the insured risk and the existence of a dependence between these factors and the occurrence of the damage. This step allows to obtain the elements of the calculation of the fair insurance premium, obtained by multiplying the estimated frequency by the estimated cost of claims. The pure premium, as the sum of the fair premium and a safety load, is justified by the results of the utility theory and by those obtained by Bruno de Finetti, in the context of the probability of ruin, with the aim of safeguarding the company from bankruptcy, as well as providing a positive expected return to the insurer. The main theoretical results are summarized with reference to these quantities in the case of a homogeneous portfolio (sum of k claims independent for the severities Z_i identically distributed) based on choice of Poisson mixed distribution for claims number as emerged by empirical evidence.

CLAIMS NUMBER: Mean, Variance and Skewness of Poisson Mixed Process

$$E(Q)=1 \quad p_k = \int k \frac{e^{-nq}(nq)^k}{k!} dF(Q)$$

Expected claims number

$$E(K)=\sum_k \int k \frac{e^{-nq}(nq)^k}{k!} dF(Q)=\sum_k \int nq \frac{e^{-nq}(nq)^{k-1}}{(k-1)!} dF(Q)=n\int q \sum_k \frac{e^{-nq}(nq)^{k-1}}{(k-1)!} dF(Q)=$$

$$=n\int q dF(q)=n$$

Second moment from origin

$$E(K^2)=\sum_k \int k^2 \frac{e^{-nq}(nq)^k}{k!} dF(Q)=\sum_k \int k \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)=\sum_k \int (k-1+1) \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)=$$

$$=\sum_k \int (k-1) \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)+\sum_k \int \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)=\sum_k \int (nq)^2 \frac{e^{-nq}(nq)^{k-2}}{(k-2)!} dF(Q)+\sum_k \int nq \frac{e^{-nq}(nq)^{k-1}}{(k-1)!} dF(Q)=$$

$$=n^2\int q^2 \sum_k \frac{e^{-nq}(nq)^{k-2}}{(k-2)!} dF(Q)+n\int q \sum_k \frac{e^{-nq}(nq)^{k-1}}{(k-1)!} dF(Q)=n^2\int q^2 dF(Q)+n\int q dF(Q)=n^2(\sigma_q{}^2+1)+n$$

Third moment from origin

$$E(K^3)=\sum_k \int k^3 \frac{e^{-nq}(nq)^k}{k!} dF(Q)=\sum_k \int k^2 \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)=\sum_k \int (k^2-k+k) \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q)=$$

$$= \sum_k \int (k^2 - k) \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q) + \sum_k \int k \frac{e^{-nq}(nq)^k}{(k-1)!} dF(Q) = \sum_k \int k \frac{e^{-nq}(nq)^k}{(k-2)!} dF(Q) + n^2(\sigma_q{}^2 + 1) + n =$$

$$= \sum_k \int (k-2+2) \frac{e^{-nq}(nq)^k}{(k-2)!} dF(Q) + n^2(\sigma_q{}^2 + 1) + n = \sum_k \int (k-2) \frac{e^{-nq}(nq)^k}{(k-2)!} dF(Q) + 2\sum_k \int \frac{e^{-nq}(nq)^k}{(k-2)!} dF(Q) +$$

$$+ n^2(\sigma_q{}^2 + 1) + n = \sum_k \int (nq)^3 \frac{e^{-nq}(nq)^{k-3}}{(k-3)!} dF(Q) + 2n^2(\sigma_q{}^2 + 1) + n^2(\sigma_q{}^2 + 1) + n =$$

$$= n^3 \int q^3 \sum_k \frac{e^{-nq}(nq)^{k-3}}{(k-3)!} dF(Q) + 3n^2(\sigma_q{}^2 + 1) + n = n^3 \int q^3 dF(Q) + 3n^2(\sigma_q{}^2 + 1) + n = n^3 E(q^3) + 3n^2(\sigma_q{}^2 + 1) + n$$

Variance of claims number

$$VAR(K) = E(K^2) - E(K)^2 = n^2(\sigma_q{}^2 + 1) + n - n^2 = n^2\sigma_q{}^2 + n$$

Skewness of claims number

$$E\left((K - E(K))^3\right) = E(K^3) + 2E(K)^3 - 3E(K^2)E(K) = n^3 E(q^3) + 3n^2(\sigma_q{}^2 + 1) + n + 2n^3 - 3n\left(n^2(\sigma_q{}^2 + 1) + n\right) =$$

$$= n^3 E(q^3) + 3n^2\sigma_q{}^2 + 3n^2 + n - 2n^3 - 3n^3\sigma_q{}^2 - 3n^3 - 3n^2 = n^3 E(q^3) - 3n^3 E(q^2) + 2n^3 + n + 3n^2\sigma_q{}^2 =$$

$$= n^3 \gamma_q \sigma_q{}^3 + n + 3n^2\sigma_q{}^2$$

$$\gamma(K) = \frac{E\left((K - E(K))^3\right)}{VAR(K)^{\frac{3}{2}}} = \frac{n^3\gamma_q\sigma_q{}^3 + n + 3n^2\sigma_q{}^2}{\left(n^2\sigma_q{}^2 + n\right)^{\frac{3}{2}}}$$

GLOBAL LOSSES: Mean, Variance and Skewness

Expected global losses

$$E(X) = \sum_k p_k E\left(\sum_{i=1}^{k} Z_i\right) = \sum_k p_k k E(Z) = E(K)E(Z)$$

Second moment from origin

$$E(X^2) = \sum_k p_k E\left[\left(\sum_{i=1}^{k} Z_i\right)^2\right] = \sum_k p_k E\left[\left(\sum_{i=1}^{k} Z_i{}^2 + \sum_i \sum_{j \neq i} Z_i Z_j\right)\right] =$$

$$= \sum_k p_k \left(\sum_{i=1}^{k} E\left(Z_i{}^2\right) + \sum_i \sum_{j \neq i} E(Z_i)E(Z_j)\right) = \sum_k p_k \left(kE(Z^2) + k(k-1)E(Z)^2\right) =$$

$$= E(K)E(Z^2) + E(K^2)E(Z)^2 - E(K)E(Z)^2$$

Third moment from origin

$$E(X^3) = \sum_k p_k E\left[\left(\sum_{i=1}^{k} Z_i\right)^3\right] = \sum_k p_k E\left[\left(\sum_{i=1}^{k} Z_i^{\,3} + 3\sum_i \sum_{j\neq i} Z_i^{\,2} Z_j + \sum_i \sum_{j\neq i} \sum_{l\neq j\neq l} Z_i Z_j Z_l\right)\right] =$$

$$= \sum_k p_k \left(\sum_{i=1}^{k} E\left(Z_i^{\,3}\right) + 3\sum_i \sum_{j\neq i} E\left(Z_i^{\,2}\right)E\left(Z_j\right) + \sum_i \sum_{j\neq i} \sum_{l\neq j\neq l} E\left(Z_i\right)E\left(Z_j\right)E\left(Z_l\right)\right) =$$

$$= \sum_k p_k \left(kE(Z^3) + 3k(k-1)E(Z^2)E(Z) + k(k-1)(k-2)E(Z)^3\right) =$$

$$= \sum_k p_k \left(kE(Z^3) + 3(k^2-k)E(Z^2)E(Z) + (k^3 - 3k^2 + 2k)E(Z)^3\right) =$$

$$= E(K)E(Z^3) + 3E(K^2)E(Z^2)E(Z) - 3E(K)E(Z^2)E(Z) + E(K^3)$$
$$E(Z)^3 - 3E(K^2)E(Z)^3 + 2E(K)E(Z)^3$$

Definitions

$$E(K) = n; E(K^2) = a_{2k}; E(K^3) = a_{3k}; VAR(K) = \sigma_k^{\,2};$$
$$E(Z) = m; E(Z^2) = a_{2z}; E(Z^3) = a_{3z}; VAR(Z) = \sigma_z^{\,2}$$

Variance

$$VAR(X) = E\left(\left(X - E(X)\right)^2\right) = E(X^2 + E(X)^2 - 2XE(X)) = E(X^2) - E(X)^2 =$$
$$= na_{2z} + a_{2k}m^2 - nm^2 - n^2m^2 = m^2\sigma_k^{\,2} + n\sigma_z^{\,2}$$

Compound Poisson Mixed Process Case

$$VAR(X) = m^2\sigma_k^{\,2} + n\sigma_z^{\,2} = m^2(n + n^2\sigma_q^{\,2}) + n(a_{2z} - m^2) = na_{2z} + n^2\sigma_q^{\,2}m^2$$

Skewness

$$E\left(\left(X - E(X)\right)^3\right) = E\left(X^3 - E(X)^3 + 3XE(X)^2 - 3X^2E(X)\right) =$$

$$= na_{3z} + 3a_{2k}a_{2z}m - 3na_{2z}m + a_{3k}m^3 - 3a_{2k}m^3 + 2nm^3 + 2n^3m^3 - 3nm(na_{2z} + a_{2k}m^2 - nm^2) =$$

$$= na_{3z} + 3a_{2k}a_{2z}m - 3na_{2z}m + a_{3k}m^3 - 3a_{2k}m^3 + 2nm^3 + 2n^3m^3 - 3n^2a_{2z}m - 3na_{2k}m^3 + 3n^2m^3 =$$

$$= na_{3z} + 3\sigma_k^{\,2}a_{2z}m + \gamma_k\sigma_k^{\,3}m^3 - 3\sigma_k^{\,2}m^3 - 3na_{2z}m + 2nm^3 = \gamma_k\sigma_k^{\,3}m^3 + n\gamma_z\sigma_z^{\,3} + 3\sigma_k^{\,2}\sigma_z^{\,2}m$$

$$\gamma(X)=\frac{E\left((X-E(X))^3\right)}{VAR(X)^{\frac{3}{2}}}=\frac{\gamma_k\sigma_k{}^3m^3+n\gamma_z\sigma_z{}^3+3\sigma_k{}^2\sigma_z{}^2}{\left(m^2\sigma_k{}^2+n\sigma_z{}^2\right)^{\frac{3}{2}}}$$

Compound Poisson Mixed Process Case

$$E\left((X-E(X))^3\right)=\gamma_k\sigma_k{}^3m^3+n\gamma_z\sigma_z{}^3+3\sigma_k{}^2\sigma_z{}^2=$$

$$=+(n+3n^2\sigma_q{}^2+n^3\gamma_q\sigma_q{}^3)m^3+na_{3z}-3na_{2z}m+2nm^3+3(n+n^2\sigma_q{}^2)a_{2z}m-3(n+n^2\sigma_q{}^2)m^3=$$

$$=nm^3+3n^2\sigma_q{}^2m^3+n^3\gamma_q\sigma_q{}^3m^3+na_{3z}-3na_{2z}m+2nm^3+3na_{2z}m+3n^2\sigma_q{}^2a_{2z}m-3nm^3-3n^2\sigma_q{}^2m^3=$$

$$=na_{3z}+3n^2\sigma_q{}^2a_{2z}m+n^3\gamma_q\sigma_q{}^3m^3$$

$$\gamma(X)=\frac{na_{3z}+3n^2\sigma_q{}^2a_{2z}m+n^3\gamma_q\sigma_q{}^3m^3}{\left(na_{2z}+n^2\sigma_q{}^2m^2\right)^{\frac{3}{2}}}$$

All the previous magnitudes could be estimated from empirical data with statistical method.

PURE PREMIUM

Expected Utility Criterium

$$E[u(P-x)]=0$$

Quadratic utility function

$$u(x)=x-\frac{x^2}{2B}$$

$$E[u(P-X)]=E\left[P-X+\frac{(P-X)^2}{2B}\right]=E\left[P-X-\frac{P^2+X^2-2PX}{2B}\right]=$$

$$=\frac{1}{2B}E\left[2BP-2BX-P^2-X^2+2PX\right]=\frac{1}{2B}\left(2BP-2BE(X)-P^2-E\left(X^2\right)+2PE(X)\right)=0$$

$$2BP-2BE(X)-P^2-E\left(X^2\right)+2PE(X)=-P^2+(2B+2E(X))P-2BE(X)-E\left(X^2\right)=0$$

$$P=\frac{-(B+E(X))\pm\sqrt{(B+E(X))^2-2BE(X)-E\left(X^2\right)}}{-1}=$$

$$=B+E(X)\pm\sqrt{B^2+E(X)^2+2BE(X)-2BE(X)-E\left(X^2\right)}=B+E(X)\pm\sqrt{B^2-\operatorname{var}(X)}$$

$$P = B + E(X) - \sqrt{B^2 - \mathrm{var}(X)} = B + E(X) - B\sqrt{1 - \frac{\mathrm{var}(X)}{B^2}} = E(X) + B\left(1 - \sqrt{1 - \frac{\mathrm{var}(X)}{B^2}}\right)$$

$$\left(1 - \sqrt{1 - \frac{\mathrm{var}(X)}{B^2}}\right) = 0 + \frac{1}{2B^2}\mathrm{var}(X) + \frac{1}{8B^4}\mathrm{var}(X)^2 + \ldots$$

$$P \approx E(X) + \frac{1}{2B}\mathrm{var}(X) + \frac{1}{8B^3}\mathrm{var}(X)^2 + \ldots \approx E(X) + \frac{1}{2B}\mathrm{var}(X)$$

Exponential utility function

$$u(x) = B\left[1 - e^{-\frac{X}{B}}\right]$$

$$E[u(P - X)] = E\left[B\left[1 - e^{-\frac{P-X}{B}}\right]\right] = B\left[1 - E\left[e^{-\frac{P-X}{B}}\right]\right] = 0$$

$$E\left[e^{-\frac{P-X}{B}}\right] = 1 \qquad \ln E\left[e^{-\frac{P-X}{B}}\right] = 0$$

$$\ln e^{-\frac{P}{B}} E\left[e^{\frac{X}{B}}\right] = -\frac{P}{B} + \ln E\left[e^{\frac{X}{B}}\right] = 0$$

$$P = B \ln E\left[e^{\frac{X}{B}}\right] \approx B\left(\frac{E(X)}{B} + \frac{\mathrm{var}(X)}{2B^2} + \ldots\right) \approx E(X) + \frac{\mathrm{var}(X)}{2B} = E(X) + m$$

Proof

$$\varphi(\lambda) = E\left(e^{-\lambda x}\right) = \int_{-\infty}^{+\infty} e^{-\lambda x} dF(x) \quad \psi(\lambda) = \varphi(-\lambda) = E\left(e^{\lambda x}\right) = \int_{-\infty}^{+\infty} e^{\lambda x} dF(x)$$

$$E\left(X^n\right) = \psi^{(n)}(0) = \lim_{\lambda \to 0} \int_{-\infty}^{+\infty} x^n e^{\lambda x} dF(x)$$

$$\ln \psi(\lambda) = \ln E\left(e^{\lambda x}\right) \qquad \ln \psi(\lambda) = \sum_{h=1}^{n} k_h \frac{\lambda^h}{h!} + o\left(\lambda^n\right)$$

$$k_h = \left.\frac{d^h \ln\psi(\lambda)}{d\lambda^h}\right|_{\lambda=0} \qquad \ln\psi^{(1)}(0) = \lim_{\lambda\to 0}\frac{\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x)}{\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)} = E(X)$$

$$\ln\psi^{(2)}(0) = \lim_{\lambda\to 0}\left[\frac{\int_{-\infty}^{+\infty} x^2e^{\lambda x}dF(x)\int_{-\infty}^{+\infty} e^{\lambda x}dF(x) - \left(\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x)\right)^2}{\left(\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)\right)^2}\right] = E(X^2) - E(X)^2 = \text{var}(X) = \sigma^2$$

$$\ln\psi^{(3)}(0) =$$

$$= \lim_{\lambda\to 0}\left[\begin{array}{l}\dfrac{\left(\int_{-\infty}^{+\infty} x^3e^{\lambda x}dF(x)\int_{-\infty}^{+\infty} e^{\lambda x}dF(x) + \int_{-\infty}^{+\infty} x^2e^{\lambda x}dF(x)\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x) - 2\int_{-\infty}^{+\infty} x^2e^{\lambda x}dF(x)\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x)\right)\left(\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)\right)^2}{\left(\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)\right)^4} + \\ -\dfrac{\left(\int_{-\infty}^{+\infty} x^2e^{\lambda x}dF(x)\int_{-\infty}^{+\infty} e^{\lambda x}dF(x) - \left(\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x)\right)^2\right)2\int_{-\infty}^{+\infty} xe^{\lambda x}dF(x)\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)}{\left(\int_{-\infty}^{+\infty} e^{\lambda x}dF(x)\right)^4}\end{array}\right] =$$

$$= E(X^3) + E(X)^2E(X) - 2E(X)^2E(X) - E(X)^2E(X) + 2E(X)^3 = E[X - E(X)]^3 = \mu_3$$

$$\ln\psi(\lambda) = E(X)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \frac{1}{3!}\gamma\sigma^3\lambda^3 + o(\lambda^3)$$

Compound Poisson Process Case

$$X = \sum_{h=0}^{N} Y_h$$

$$E\left[e^{\frac{X}{B}}\right] = E\left\{E\left[e^{\frac{\sum_{h=0}^{N}Y_h}{B}}\middle| N\right]\right\} = E\left\{E\left[\prod_{h=0}^{N} e^{\frac{Y_h}{B}}\middle| N\right]\right\} = \sum_n\left[E\left(e^{\frac{Y}{B}}\right)\right]^n prob\{N = n\} = E(t^N)$$

An important criticism of the pricing approach through risk classification is given by the impossibility of integrating some information on the policyholder into the calculation of the premium. This information cannot be captured by the insurer and may represent significant risk factors. In this context, the actuarial practice introduces the a posteriori phase in the pricing process, based on the theory of credibility. In this way, the policyholders are evaluated through the history of the individual, inserting the retrospective component in the calculation of the insurance premium. In other words, the analysis of a posteriori premiums allows the correction and adjustment of an a priori premium to obtain a reasonable forecast.

The a priori and a posteriori characteristics of a risk are well summarized in two fun images selected by the Norwegian actuary Haavardsson:

……will which house be likely burst into flames??

Figure 2: Risk selection: objective risks – a priori

……….. sloppy and unlucky client

Figure 3: Risk selection: subjective risks – a posteriori

A practical version of this theory is the bonus-malus system in motor liability insurance, through which the insured's past experience with regard to risk generation is considered. If, during the a priori pricing phase, insurers have the freedom to set premiums based on adequate and relevant risk factors, the bonus-malus system is required by the law of each country and must be respected by each insurance company, sometimes without any changes in its implementation. The bonus-malus system is intended both as an integrated part of the premium intended to combat the problem of information asymmetry, and as a means of incrementing (if free) competition between insurance companies.

On these coordinates, the main issue of research is the construction and analysis of econometric models to estimate the frequency and cost of compensation, based on the information available. The applicability of actuarial science to non-life insurance purposes has a rich and long history. There are dedicated institutions, such as the Casualty Actuarial Society (CAS) Institution of USA which was founded in 1914. Therefore, analysing the literature, one can observe over time the exceptional contribution of researchers to understand the specifications and functionality of the actuarial methods, as well as the effort of actuaries to adapt and develop new models for risk assessment taking into account the requirements and challenges of the evolving insurance market.

In order to determine effective insurance premiums to combat information asymmetry, the insurance portfolio is divided into subportfolios in which the risks can be considered homogeneous. This leads to the definition of the risk classes that will have assigned a different premium according to the correspondent severity of the risks. In this regard, an important aspect is underlined by the risk classification criteria. Therefore, if the risks are grouped based on a priori information regarding the policyholders or the insured assets, the groups obtained are called "a priori". On the contrary, if information on the claims history of each insured is taken into consideration, "a posteriori" risk classes are outlined. Considering this distinction, the actuarial literature refers to two concepts of pricing, namely a priori and a posteriori. By applying various actuarial techniques, corresponding to each type of pricing, the aim is to find an integrated solution that makes it possible to establish a fair premium based on the nature of the risks and classification.

2.2. Empirical approach to *pricing*

In the eighteenth century, in fire insurance, the premiums were based on the type of roof and the structure of the buildings and for maritime insurance, considered the oldest form of insurance, the premium was based on the design characteristics of the ship. Considering the presence of uncertain events that may occur depending on certain risk factors, the actuaries have always tried to find a mathematical formulation in order to determine the probability of risk occurrence and to establish the insurance premium.

The involvement of actuarial science in the insurance industry has a long and rich history and is based on the mathematical theory of risk.

Under the considerable influence of the studies of Lundberg (1903) and Cramer (1930), considered the founders of the theory of risk, actuaries have always been interested in treating risks from the perspective of insurance companies.

The monograph published by Hans Bühlmann (1970) [4] highlights the recognition of actuarial mathematics as a fundamental topic in probability theory and statistics applied to non-life insurance. As mentioned by Hans Gerber, the determination of the probability law of the loss cost has always been the central argument in the literature on risk theory.

Analysing it retrospectively, at the beginning actuarial science was limited to the use of linear Gaussian models, assuming the use of regression analysis that aimed to quantify the impact of the explanatory variables on the phenomenon of interest. The linear model, proposed by Legendre and Gauss in the nineteenth century, played a crucial role in econometrics, but the applicability of this model to insurance was judged difficult. In this context, linear modelling implies a series of hypotheses (Gaussian probability density, linearity of the predictor and homoscedasticity) that are not compatible with the reality imposed by the frequency and cost of the damage generated by the risky event.

As the complexity of the statistical criteria became more pronounced, the actuaries had to solve the problem of finding some models that explain the occurrence of the risk as realistically as possible. Although no mathematical model will ever completely describe reality, the analysis of the models and the comparison between the theoretical properties of the phenomenon studied and the observed one is a pragmatic way to acquire a better understanding of reality and to predict the future responses of the events analysed.

2.3. Pricing a priori

The fundamental idea of a priori pricing is to segment the insured risks into different categories so that within each category the risks are considered equivalent and grouped according to the same criterion.

According to Delaporte (1972) [7], a priori pricing makes it possible to group risks into tariff classes, including in each group the policyholders with the same risk profile who will pay the same premium. The first milestone of a priori pricing in non-life insurance is considered the minimum bias risk classification procedure proposed by Bailey and Simon (1960) [1] and Bailey (1963) [2]. This method uses an iterative algorithm to calculate the optimal values of each risk level by minimizing the error function. Although it was configured outside a recognized statistical framework, the actuarial literature demonstrates that this iterative "heuristic" approach is a special case of generalized linear models (GLM).

Starting from the actuarial definition of McCullagh and Nelder (1989) [11], GLMs have become common practice in the statistical sector for the pricing of non-life insurance. The two authors pointed out two major advantages for GLMs. First, the generalization of linear modelling allows us to overcome the assumption of normality, since the regression is extended to the class of the exponential family (Normal, Poisson, Binomial and gamma). Secondly, GLMs allow linear regression to be related to the dependent variable through the link function, modelling the additive effect of independent variables on a transformation of the mean, instead of the mean itself. In other words, this function is the linear predictor or the score at the mean of the dependent variable. Compared to minimum bias procedure techniques, GLM models have the advantage of providing a theoretical framework that allows the use of statistical tests to evaluate the adaptation of the models.

In the actuarial literature, Jean Lemaire (1985) [9] distinguished himself by illustrating and measuring the effectiveness of the methods used to estimate the insured risks in the motor third-party liability line of business, selecting the explanatory variables. In addition, the book by Ohlsson and Johansson (2010) [13] comprehensively dealt with the methods considered as the basis in the classification of insurance risk, with particular attention to statistical techniques for calculating the motor insurance premium.

2.4. Pricing a posteriori

Pricing a priori alone involves the lack of fundamental elements on the connection between some tariff variables and the risk injured. Certain important risk factors may not be observed or unobservable, leading to the consequence of violation of the hypothesis of homogeneity, which is essential for an effective risk classification system. For going beyond the limits of this type of pricing, the approach of a posteriori actuarial models that take into account additional information on the history of the contracting party’s claims can be adopted.

A posteriori pricing is based on credibility theory. Savage (1954) [16] points out that the notion of credibility is closely related to the perception of risk, as individuals assign different degrees of credibility to the occurrence of certain events. Savage also discusses the degree of belief, this notion being first introduced by Thomas Bayes (1763) in his essay on the doctrine of probabilities. Although the concept of credibility was systematized in the middle of the 20th century, since 1910, employees of General Motors who were insured against accidents at work benefited from a premium calculated according to this principle, later formalized by Mowbray (1914) [12]. In this case, the credibility theory only admits two levels, one and zero.

This situation means, for an employee just below the eligibility threshold, a significant difference in the premium he has to pay. To respond to this criticism, Whitney (1918) [17] introduces the concept of partial credibility, arguing that the problem of evaluating experience arises from the need to find a balance between collective experience, on the one hand, and individual risk experience on the other hand. Therefore, Whitney states that the basic principle of credibility is to establish a weighting factor, emphasizing the definition of pure premium as a balance between the experience of a single risk and that of a risk class.

Hans Bühlmann (1967) [3] solves the problem of finding an optimal estimate for the premium corresponding to the nth period, taking into account the observations relating to the risks recorded in previous periods; manages to revolutionize the theory of credibility by introducing a credibility factor. Starting from these concepts, Bühlmann (1970) develops together with Erwin Straub, the famous Bühlmann-Straub model, whose main improvement made to the initial model is the definition of the estimators of the structural parameters. Most of the principles of credibility theory align with the basic model proposed by Bühlmann, around which all other models accepted in this area represent a generalization of Buhlamnn's idea.

The Credibility Theory is an approximate (linear) version of the Bayesian approach that coincides exactly if the observations have distribution in the exponential family and if the a priori distributions belong to the conjugates.

Although credibility theory can be seen as the art of combining different data collections to obtain an accurate global estimate, its methods are difficult to implement in practice due to their mathematical complexity. Therefore, insurance companies have used some methods, which are simplified versions of those suggested by the credibility theory. In this sense, one version of the credibility theory is the aforementioned bonus-malus system introduced by Pesonen (1962) [14] who tried to establish the rules for obtaining optimal rewards for each risk class based on the levels bonus Malus.

The basic idea of this system has been described in depth by Lemaire (1995) [10]. The bonus-malus system is defined as a scale consisting of a finite number of levels. The policyholders receive a certain place based on the transition rules and the number of claims reported. Each level corresponds to a certain coefficient that will be applied to the pure premium calculated in the a priori analysis phase. Bonus-malus systems allow premiums to be tailored to individual hidden risk factors taking into account the record of past claims. Therefore, in the context of the insurance markets, the main purpose of the bonus-malus system is to equitably assess the individual degree of risk so that the insurance company requests a premium corresponding to both the insured risk profile and the history of claims.

It should be remembered that in order to always be efficient and balanced, these systems must be monitored and possibly revised through dynamic assessments using tools such as Markov chains to see how the policyholders, given a certain claim, are distributed over time between the merit classes.

2.5. New frontiers

New artificial intelligence techniques have recently been applied to pricing processes. A recent article published in The actuary entitled "Are actuaries competitive in data science?" recalls the definition of the data scientist. He is an individual characterized by the competence deriving from the intersection of three different skills: coding / programming to manipulate data and create algorithms; mathematics and statistics to use the data for future predictions; knowledge domain to understand and manage practical business problems.

However, author Colin Priest states that while actuaries know a lot about insurance laws and regulations, underwriting, claims handling and product design, and have a very specific mathematical and statistical education, they are forced to learn programming. In the United States, traditional actuaries are being replaced by data scientists with no specific insurance knowledge to meet current business needs. So actuaries, for the purposes of full employment, must prepare to:

- Data manipulation and construction of tables;
- Theory of machine learning (training versus testing, overtraining);
- Machine Learning algorithms;
- Mathematics and Statistics: imputation of missing values, optimization and numerical estimates.

The good news is that modern technology makes this easier than in the past. With new technology tools, actuaries can become more competitive in data science and, in addition to their knowledge of the insurance business and communication skills, they can have a competitive advantage over other resources.

3. A PROPOSAL FOR CONSISTENCY OVER TIME

Here a proposal to monitor the consistency of the personalized premiums over time in a simplified approach is presented. It is important to verify the consistency between actual premiums with technical basis especially for mandatory motor third liability insurance as required by some National legislations.

First, the commercial coefficients are considered acceptable if they do not significantly subvert the ordering of relativities (coefficients) resulting from the technical analysis. Otherwise, they would invalidate the correct measurement of the personalization of risks that is supposed to be contained in the tariff.

The differences between the technical and commercial coefficients must also be evaluated also in relation to the effects on the difference between the theoretical personalized bonus and the one applied. The transmission channel must be identified in the correction mechanism of the "reference premium" (or basis for the application of the customization coefficients) necessary - following the changes induced by the company - to guarantee the tariff requirement.

For example, if a variable, such as the territorial area, was divided into twenty classes and a company wanted to impose the reduction of the tariff coefficients by half of the same classes while keeping those of the others constant, this operation would not be neutral on the final premiums. In fact, the company, in order to still guarantee the requirement, will have to increase the reference premium equal for all classes (calculated on the basis of the distribution of policyholders and all tariff coefficients). After the application of the coefficients, the effect obtained will be a reduction (slightly more contained than that established by the company on the coefficients) of the personalized premiums for ten territorial classes and an increase in the premiums for the other ten whose corresponding coefficients have not been changed.

This phenomenon of relative movement implies a further reflection that allows deriving a different judgment depending on the directions given to the tariff changes. An example with two stylized cases may help, taking into account that in reality the following mechanisms are usually amplified:

i) Taking two policyholder profiles, the first with a high propensity to claim claims and the other with a low propensity, a significant increase in the personalization coefficient of the first or a reduction, also significant, of that of the second, induces an excessive personalization such as to unbalance the relationships between the technical bonuses;

ii) vice versa, a reduction in the personalization coefficient of the high claims profile or an increase in the low profile induces a solidarity correction that could be considered acceptable if it does not violate the principle of consistency of the premiums with the technical bases.

Lastly, the absolute level assumed by the tariff premiums must also always be monitored.

A method that could be adopted to carry out the analyses could consist in preparing a special prospectus to be used to evaluate the company's choices. This table would report the rate premiums relating to profiles resulting from significant combinations of the main rate variables of "strategic" interest for the company. Furthermore, these bonuses must actually be neutralized with respect to the distributions in the classes of the other rate variables used by the company.

The combinations of interest must take into account:

- Profiles representative of a number of persons exposed to significant (insured) risk;
- Classes characterized by potentially very loss-making risks and therefore by high premiums (of which it is essential to highlight the level assumed) even in correspondence with a limited number of policyholders.

The comparison must be made between the tariff premiums applied and those resulting from the technical-actuarial analysis before the commercial interventions defined by the company. The examination of the prospectus should be focused on the evaluation of these deviations.

The synthetic variation of the difference in the product between the modified technical coefficients and the commercial ones must in any case be analysed for each selected profile as it is useful for formulating the judgment of technical consistency of the tariff as a whole, even in the presence of tariff adjustments established by the company.

The joint analysis of the alerts - established by the actuary - highlighted for all profiles and a measurement of the corresponding incidence, carried out with due diligence, will support the actuary in providing a summary judgment on the rate applied.

Theoretical framework step by step

a) Average actual premium

$$P^M = P^E(1+q) = f\bar{c}(1+q) = \frac{n}{r}\frac{S}{n}(1+q)$$

P^M average actual premium

P^E pure premium

q general expenses charge

f expected claims frequency

$\bar{c}$ expected average claims cost

n expected claims number

r risks-year exposure

S incurred claims forecast

b) Explanatory variables and related personalization coefficients

$\mathrm{A}, \mathrm{B}, \Gamma, \Delta, \mathrm{E}, \mathrm{Z}$ explanatory variables

$\alpha_1,\ldots,\alpha_i,\ldots,\alpha_I$

$\beta_1,\ldots,\beta_j,\ldots,\beta_J$

$\gamma_1,\ldots,\gamma_k,\ldots,\gamma_K$

$\delta_1,\ldots,\delta_l,\ldots,\delta_L$

$\varepsilon_1,\ldots,\varepsilon_m,\ldots,\varepsilon_M$

$\zeta_1,\ldots,\zeta_n,\ldots,\zeta_N$

In the proposed framework:

- The first three define the general **profile** (e.g. geographic zone, age, bonus-malus class);
- The fourth is a priori fixed for the profile (e.g. modal class of variable cubic cilindrate, minimum loss limit) or determinated consequently (e.g. Age of driving license for a young);
- The fifth and the sixth represent chosen variables from undertaking that do not define the tariff profile (es. occupancy, vehicle age) and have to be neutralized.

r_{ijklmn} Risks for year of the generic tariff class generated from the intersection of the six variables

c) Reference premium (technical)

$$P^{R(T)} = \frac{P^M}{\frac{1}{r}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn}} = \frac{P^M}{\pi}$$

$P^{R(T)}$ reference premium (technical)

$$\pi = \frac{1}{r}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn}$$

π average premium coefficient (technical)

d) Personalized premium for a specific tariff class

$\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}$ tariff class

$P^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}(T)} = P^{R(T)}\alpha_{\bar{i}}\beta_{\bar{j}}\gamma_{\bar{k}}\delta_{\bar{l}}\varepsilon_{\bar{m}}\zeta_{\bar{n}}$ personalized premium of the class (technical)

e) Budget Equilibrium

$$\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} P^{ijklmn} r_{ijklmn} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} P^{R(T)}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn} =$$

$$= P^{R(T)}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn} = \frac{P^M\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn}}{\frac{1}{r}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N}\alpha_i\beta_j\gamma_k\delta_l\varepsilon_m\zeta_n r_{ijklmn}} = rP^M$$

f) Personalized premium (technical) for a specific profile

$\bar{i}\bar{j}\bar{k}$ specific class intersection $\bar{l}$ determined variable

$$P^{\bar{i}\bar{j}\bar{k}\bar{l}(T)} = P^{R(T)}\alpha_{\bar{i}}\beta_{\bar{j}}\gamma_{\bar{k}}\delta_{\bar{l}}\frac{1}{r}\sum_{m=1}^{M}\sum_{n=1}^{N}\varepsilon_m\zeta_n r_{mn}$$

$P^{\bar{i}\bar{j}\bar{k}\bar{l}(T)}$ personalized premium of the profile (technical)

g) Enterpreneur decisions on tariff coefficients

$a_1,\ldots,a_i,\ldots,a_I$

$b_1,\ldots,b_j,\ldots,b_J$

$g_1,\ldots,g_k,\ldots,g_K$

$d_1,\ldots,d_l,\ldots,d_L$

$e_1,\ldots,e_m,\ldots,e_M$

$z_1,\ldots,z_n,\ldots,z_N$

h) Reference premium (commercial)

$$P^{R(C)} = \frac{P^M}{\frac{1}{r}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} a_i b_j g_k d_l e_m z_n r_{ijklmn}} = \frac{P^M}{p}$$

$P^{R(C)}$ reference premium (commercial)

$$p = \frac{1}{r}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}\sum_{l=1}^{L}\sum_{m=1}^{M}\sum_{n=1}^{N} a_i b_j g_k d_l e_m z_n r_{ijklmn}$$

p average premium coefficient

i) Personalized premium for a specific tariff class

$\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}$ tariff class

$P^{\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}\bar{n}(C)} = P^{R(C)} a_{\bar{i}} b_{\bar{j}} g_{\bar{k}} d_{\bar{l}} e_{\bar{m}} z_{\bar{n}}$ personalized premium (commercial)

j) Personalized premium (commercial) for a specific profile

$\bar{i}\bar{j}\bar{k}$

$$P^{\bar{i}\bar{j}\bar{k}\bar{l}(C)} = P^{R(C)} a_{\bar{i}} b_{\bar{j}} g_{\bar{k}} d_{\bar{l}} \frac{1}{r}\sum_{m=1}^{M}\sum_{n=1}^{N} e_m z_n r_{mn}$$

$P^{\bar{i}\bar{j}\bar{k}\bar{l}(C)}$ Personalized premium (commercial) for a specific profile

k) Indicator functions for the adjustment

$$\rho_i^{[1]} = \begin{cases} 0 & a_i = \alpha_i \\ 1 & a_i \neq \alpha_i \end{cases} \qquad \rho_j^{[2]} = \begin{cases} 0 & b_j = \beta_j \\ 1 & b_j \neq \beta_j \end{cases} \qquad \rho_k^{[3]} = \begin{cases} 0 & g_k = \gamma_k \\ 1 & g_k \neq \gamma_k \end{cases}$$

$$\rho_l^{[4]} = \begin{cases} 0 & d_l = \delta_l \\ 1 & d_l \neq \delta_l \end{cases} \qquad \rho_m^{[5]} = \begin{cases} 0 & e_m = \varepsilon_m \\ 1 & e_m \neq \varepsilon_m \end{cases} \qquad \rho_n^{[6]} = \begin{cases} 0 & z_n = \zeta_n \\ 1 & z_n \neq \zeta_n \end{cases}$$

l) Profile analysis

$\bar{i}\bar{j}\bar{k}$

1) Criterion of red warning

$P^{\bar{i}\bar{j}\bar{k}\bar{l}(C)} > kP^M$ $\qquad$ $P^{\bar{i}\bar{j}\bar{k}\bar{l}(T)} > kP^M$

(e.g. $k = 5$)

k moltiplicator of actual average premium

2) Criterion of yellow warning

$$\frac{P^{\bar{i}\bar{j}\bar{k}\bar{l}(C)} - P^{\bar{i}\bar{j}\bar{k}\bar{l}(T)}}{P^{\bar{i}\bar{j}\bar{k}\bar{l}(T)}} > t_{yellow}$$

(e.g. $t_{green} > 15\%$)

3) Criterion of green warning

$$\frac{a_{\bar{i}} b_{\bar{j}} g_{\bar{k}} d_{\bar{l}} \rho_i^{[1]} \rho_j^{[2]} \rho_k^{[3]} \rho_l^{[4]} \frac{1}{r} \sum_{m=1}^{M} \sum_{n=1}^{N} e_m z_n r_{mn} - \alpha_{\bar{i}} \beta_{\bar{j}} \gamma_{\bar{k}} \delta_{\bar{l}} \rho_i^{[1]} \rho_j^{[2]} \rho_k^{[3]} \rho_l^{[4]} \frac{1}{r} \sum_{m=1}^{M} \sum_{n=1}^{N} \varepsilon_m \zeta_n r_{mn}}{\alpha_{\bar{i}} \beta_{\bar{j}} \gamma_{\bar{k}} \delta_{\bar{l}} \rho_i^{[1]} \rho_j^{[2]} \rho_k^{[3]} \rho_l^{[4]} \frac{1}{r} \sum_{m=1}^{M} \sum_{n=1}^{N} \varepsilon_m \zeta_n r_{mn}} > t_{green}$$

(e.g. $t_{green} > 30\%$)

The colour from red to green give a different magnitude on intervention needed to the actual tariff.

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